Characterization of symmetric M-matrices as resistive inverses

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Abstract

We aim here at characterizing those nonnegative matrices whose inverse is an irreducible Stieltjes matrix. Specifically, we prove that any irreducible Stieltjes matrix is a resistive inverse. To do this we consider the network defined by the off-diagonal entries of the matrix and we identify the matrix with a positive definite Schrödinger operator which ground state is determined by the lowest eigenvalue of the matrix and the corresponding positive eigenvector. We also analyze the case in which the operator is positive semidefinite which corresponds to the study of singular irreducible symmetric M-matrices.

Keywords: M-matrices, Schrödinger operators, Green kernels, Moore-Penrose Inverse, effective resistance, Kirchhoff index.

1 Introduction

In view of their numerous applications, for instance in Numerical Methods, Probability Theory and Economics, M-matrices have deserved a great attention and many of their properties has been studied. An important problem related with M-matrices is the so-called *inverse M-matrix problem*, that consists in characterizing all nonnegative matrices whose inverses are M-matrices. This is a longstanding and difficult problem that has generated a big amount of literature and has been partially solved. M. Martínez *et al.* in their celebrated paper [14], see also [16], proved that the inverse of any strictly ultrametric matrix is a diagonally dominant Stieltjes matrix and C. Dellacherie *et al.* in [6] extend this result by proving that the inverse of any nonsingular ultrametric matrix is a weakly diagonally dominant Stieltjes matrix. Two years later, M. Fiedler characterized in [8] this type of matrices as *resistive inverses* associated with networks. Specifically, if M is an irreducible weakly diagonally dominant Stieltjes matrix of order n, then there exists a connected network with $n+1$ vertices such that $\mathsf{M}^{-1} = (g_{ij})$ where $g_{ij} = \frac{1}{2}\left(R_{i,n+1} + R_{j,n+1} - R_{ij}\right)$ and (R_{ij}) is the resistance matrix of the network. The above mentioned works have generated a great amount of generalizations, see for instance [9] and [15, 17] for the non-symmetric case.

In this work we extend Fiedler's characterization to the case of irreducible Stieltjes matrices by removing the diagonally dominance hypothesis. The key idea is to identify any irreducible Stieltjes matrix with a positive definite Schrödinger operator on a suitable connected network and to take advantage of the previous work developed by the authors, [4, 5]. In this framework we generalized the concept of effective resistance and prove that such general effective resistances verify properties that are analogues to those verified by the standard effective resistances. In particular, they determine a distance on the network and hence they are of potential application specially in Chemistry, [19]. Moreover, we give a formula for the inverse of the resistance matrix that generalized the known formula for the standard case, see [2]. In this context, the relation between effective resistances and generalized inverses of the combinatorial laplacian have been widely studied, [1, 2, 10, 11, 12, 13, 18]. Here, we pay special attention to a specific kind of generalized inverses of positive semidefinite Schrödinger operators, namely the so-called Green operators and mainly to the one identified with the Moore-Penrose inverse of a singular, irreducible and symmetric M-matrix.

2 Preliminaries

Given a finite set V, the set of real valued functions on V is denoted by $\mathcal{C}(V)$. In particular, for any $x \in V$, $\varepsilon_x \in \mathcal{C}(V)$ stands for the Dirac function at x. The standard inner product on $\mathcal{C}(V)$ is denoted by $\langle \cdot, \cdot \rangle$ and hence if $u, v \in \mathcal{C}(V)$ then $\langle u, v \rangle = \sum\limits_{x \in V} u(x)\, v(x)$.

A function $K \colon V \times V \longrightarrow \mathbb{R}$ is called *a kernel on* V and determines an endomorphism of $\mathcal{C}(V)$ by assigning to any $u \in \mathcal{C}(V)$ the function $\mathcal{K}(u) = \sum\limits_{y \in V} K(\cdot, y)\, u(y)$. Conversely, each endomorphism of $\mathcal{C}(V)$ is determined by the kernel given by $K(x,y) = \mathcal{K}(\varepsilon_y)(x)$ for any $x, y \in V$. The kernel of $\mathcal{K}^*$, the *adjoint of* $\mathcal{K}$, is given by $K^*(x,y) = K(y,x)$ for any $x, y \in V$ and hence $\mathcal{K}$ is self-adjoint iff its kernel is a symmetric function.

Throughout the paper we will make use of two special endomorphisms of $\mathcal{C}(V)$, namely multiplication by a given function and projections. Specifically, given $\tau \in \mathcal{C}(V)$, we denote by $\mathcal{D}_\tau$ the endomorphism of $\mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function $\mathcal{D}_\tau(u) = \tau\, u$, whereas given $\omega, \tau \in \mathcal{C}(V)$ we denote by $\mathcal{P}_{\omega,\tau}$ the endomorphism of $\mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function $\mathcal{P}_{\omega,\tau}(u) = \langle \tau, u \rangle\, \omega$. In particular, when $\omega \neq 0$ and $\tau = \omega\, \langle \omega, \omega \rangle^{-1}$, the above endomorphism is denoted simply by $\mathcal{P}_\omega$ and clearly satisfies that $\mathcal{P}_\omega(\omega) = \omega$. So, the kernel of $\mathcal{D}_\tau$ is given by $D(x,x) = \tau(x)$ and $D(x,y) = 0$ when $y \neq x$ and the kernel of $\mathcal{P}_{\omega,\tau}$ is $\omega \otimes \tau$.

If $|V| = n$ and we suppose that the elements of V are labelled from 1 to n, then kernels, and hence its associated endomorphisms, can be identified with the matrices of order n, whereas functions on V can be alternatively identified with (column) vectors of $\mathbb{R}^n$ or diagonal matrices. In particular, the matrix identified with the identity operator, $\mathcal{I}$, is the identity matrix I, whereas if $\omega \in \mathcal{C}(V)$ the matrices identified with $\mathcal{D}_\omega$ and $\mathcal{P}_\omega$ and the vector identified with ω are denoted by D_ω, P_ω and w, respectively. Moreover, an endomorphism $\mathcal{K}$ of $\mathcal{C}(V)$ is identified with a symmetric *Z-matrix* iff its kernel, K, is symmetric and verifies that $K(x,y) \leqslant 0$ for any $x, y \in V$ with $x \neq y$. Moreover $\mathcal{K}$ is identified with a symmetric *M-matrix*, respectively an *Stieltjes matrix*, iff in addition it is positive semidefinite, respectively positive definite.

In this work we mainly use the terminology of endomorphisms of $\mathcal{C}(V)$ and their kernels. We preferred to do this because then we do not need to label the elements of V and also because our methodology appears as the discrete counterpart of the standard treatment of resolvent operators on Riemannian manifolds. On the other hand, if we consider a symmetric Z-matrix of order n with diagonal null entries and a its associated kernel, then $c = -a$ can be seen as the conductance function of a network whose vertex set is V and moreover the matrix is irreducible iff the network is connected, see below for definitions.

The triple $\Gamma = (V, E, c)$ denotes a finite network; that is, a finite connected graph without loops nor multiple edges, with vertex set V, whose cardinality equals n, and with edge set E, in which each edge $\{x, y\}$ has been assigned a *conductance* $c(x, y) > 0$. So, the conductance can be considered as a symmetric kernel $c\colon V \times V \longrightarrow [0, +\infty)$ such that $c(x, x) = 0$ for any $x \in V$ and moreover, vertex x is adjacent to vertex y iff $c(x, y) > 0$. Definitely, a finite network is entirely characterized by its vertex set and its conductance kernel and hence in the sequel it will be represented as $\Gamma = (V, c)$. Given $x, y, z \in V$, we say that *z separates x and y* iff the set $V \setminus \{z\}$ is not connected and x and y belong to different connected components.

The *combinatorial Laplacian* or simply the *Laplacian* of the network Γ is the endomorphism of $\mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function

$$\mathcal{L}(u)(x) = \sum_{y \in V} c(x, y) \Big(u(x) - u(y) \Big), \quad x \in V. \tag{1}$$

It is well-known, that the Laplacian is a self-adjoint positive semidefinite operator and moreover $\mathcal{L}(u) = 0$ iff u is a constant function.

Definition 2.1 *Given $q \in \mathcal{C}(V)$, the Schrödinger operator on Γ with ground state q is the endomorphism of $\mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function $\mathcal{L}_q(u) = \mathcal{L}(u) + qu$.*

The properties of the matrices identified with Schrödinger operators are described in the following result, whose proof is straightforward.

Proposition 2.2 *If the vertices of Γ are labelled from 1 to n, the set of Schrödinger operators on Γ is identified with the set of irreducible symmetric Z-matrices of order n whose off-diagonal* *elements are given by $-c$. Moreover one of these matrices is an M-matrix, respectively a Stieltjes matrix, iff the corresponding Schrödinger operator is positive semidefinite, respectively positive definite. In addition, such a matrix is a weakly diagonally dominant M-matrix iff the ground state of its corresponding Schrödinger operator is nonnegative.*

After the above proposition it is clear that it will be useful to characterize all positive semidefinite Schrödinger operators on Γ, because this will mean to characterize those irreducible and symmetric M-matrices whose off-diagonal entries are given by the conductance c. In [4], some of the authors answered this question by using a *Doob h-transform*, a very common technique in the framework of Dirichlet forms and Markov Chains.

Definition 2.3 *If $\omega(x) > 0$ for any $x \in V$, the function $q_\omega = -\dfrac{1}{\omega}\,\mathcal{L}(\omega)$ is named the ground state determined by ω.*

If ω and μ are positive functions, then $q_\mu = q_\omega$ iff $\mu = a\omega$ for some $a > 0$, see [4]. In particular, $q_\omega = 0$ iff ω is a positive constant. More generally, $\langle \omega, q_\omega \rangle = 0$, which implies that q_ω takes positive and negative values, except when ω is constant. Moreover, for any proper subset $F \subset V$ it is possible to choose a positive function ω such that $q_\omega(x) < 0$ for any $x \in F$, see [4].

As we have seen, the ground state q_ω determines ω up to a multiplicative positive constant. Although this lack of uniqueness is not important for most of the results in this work, we introduce the following definition to avoid it.

Definition 2.4 *A function $\omega \in \mathcal{C}(V)$ is called a weight if $\omega(x) > 0$ for any $x \in V$ and moreover $\langle \omega, \omega \rangle = n$. The set of weights on V that verify the above property is denoted by $\Omega(V)$.*

Observe that the only constant function in $\Omega(V)$ is given by $\omega(x) = 1$ for any $x \in V$ and it is denoted by 1. Clearly, each ground state of the form q_ω characterizes its corresponding weight. Under the above terminology the characterization of positive semidefinite Schrödinger operators is given by the following result, see [4, Proposition 3.3].

Proposition 2.5 *The Schrödinger operator $\mathcal{L}_q$ is positive semidefinite iff there exist $\omega \in \Omega(V)$ and $\lambda \geqslant 0$ such that $q = q_\omega + \lambda$. Moreover, ω and λ are uniquely determined. In addition, $\mathcal{L}_q$ is not positive definite iff $\lambda = 0$ and then, $\langle \mathcal{L}_{q_\omega}(v), v \rangle = 0$ iff $v = a\omega$, $a \in \mathbb{R}$. In any case λ is the lowest eigenvalue of $\mathcal{L}_q$ and its associated eigenfunctions are multiple of ω.*

If the vertices of Γ are labelled from 1 to n, we represent by $\mathcal{M}(c)$ and by $\mathcal{S}(c)$ the sets of irreducible symmetric M-matrices and Stieltjes matrices respectively, whose off-diagonal entries are given by $-c$. Moreover, given $\omega \in \Omega(V)$, we consider L_ω the matrix in $\mathcal{M}(c)$ identified with $\mathcal{L}_{q_\omega}$, that when $w = 1$ is denoted simply as L. So, the above proposition can be translated into the language of matrices as follows.

Corollary 2.6 *If the vertices of V are labelled from 1 to n, then*

$$\mathcal{M}(c)=\{\mathsf{L}_\omega+\lambda\mathsf{I}:\lambda\geqslant 0,\ \omega\in\Omega(V)\}\quad\textit{and}\quad \mathcal{S}(c)=\{\mathsf{L}_\omega+\lambda\mathsf{I}:\lambda>0,\ \omega\in\Omega(V)\}.$$

Moreover, for any $\lambda\geqslant 0$ and $\omega\in\Omega(V)$, $\mathsf{L}_\omega+\lambda\mathsf{I}$ is the unique M-matrix whose off-diagonal elements are given by $-c$ such that λ is its lowest eigenvalue and ω is an associated eigenvector.

The proof of the main result in this paper is based in a commonly used technique in the context of electrical networks and Markov Chains, that in fact is used in [6, 7]. We remark that in the probabilistic context, the function q is usually called *the potential (vector)* of the operator $\mathcal{L}_q$, see for instance [7]. Given $\mathcal{L}_q$ a positive definite Schrödinger operator on Γ, the method consists in embedding the given network into a suitable host network. The new network is constructed by adding a new vertex, that represents an absorbing state, joined with each vertex in the original network through a new edge whose conductance is the *diagonal excess* after the use of the h-transform.

Definition 2.7 *Given $\lambda>0$, $\omega\in\Omega(V)$ and $\hat{x}\notin V$, we consider the network $\Gamma_{\lambda,\omega}=(V\cup\{\hat{x}\},c_{\lambda,\omega})$ where $c_{\lambda,\omega}(x,y)=c(x,y)$ when $x,y\in V$ and $c_{\lambda,\omega}(\hat{x},x)=c_{\lambda,\omega}(x,\hat{x})=\lambda\,\omega(x)$ for any $x\in V$. We denote by $\mathcal{L}^{\lambda,\omega}$ its combinatorial Laplacian and by $\widehat{\omega}\in\Omega(V\cup\{\hat{x}\})$ the weight given by $\widehat{\omega}(x)=\omega(x)$ when $x\in V$ and $\widehat{\omega}(\hat{x})=1$.*

The next result establishes the relationship between the original Schrödinger operator $\mathcal{L}_q$ and a new semidefinite Schrödinger operator on $\Gamma_{\lambda,\omega}$.

Proposition 2.8 *If $q=q_\omega+\lambda$ and we define $\hat{q}=-\dfrac{1}{\widehat{\omega}}\mathcal{L}^{\lambda,\omega}(\widehat{\omega})$, then $\hat{q}(\hat{x})=\lambda\,(n-\langle\omega,1\rangle)$ and $\hat{q}=q-\lambda\,\omega$ on V. Moreover, for any $u\in\mathcal{C}(V\cup\{\hat{x}\})$ we get that $\mathcal{L}^{\lambda,\omega}_{\hat{q}}(u)(\hat{x})=\lambda\,(n\,u(\hat{x})-\langle\omega,u_{|_V}\rangle)$ and*

$$\mathcal{L}^{\lambda,\omega}_{\hat{q}}(u)=\mathcal{L}_q(u_{|_V})-\lambda\,\omega\,u(\hat{x})=\mathcal{L}_q(u_{|_V})-\lambda\,\mathcal{P}_\omega(u_{|_V})-\frac{\omega}{n}\mathcal{L}^{\lambda,\omega}_{\hat{q}}(u)(\hat{x})\quad\textit{on } V.$$

Proof. Given $u\in\mathcal{C}(V\cup\{\hat{x}\})$, then for any $x\in V$ we get that

$$\mathcal{L}^{\lambda,\omega}(u)(x)=\mathcal{L}(u_{|_V})(x)+\lambda\,\omega(x)u(x)-\lambda\,\omega(x)u(\hat{x}).$$

In particular, tacking $u=\hat{\omega}$ we obtain that $-\hat{q}=-q+\lambda\,\omega$ on V and hence

$$\mathcal{L}^{\lambda,\omega}_{\hat{q}}(u)=\mathcal{L}_q(u_{|_V})-\lambda\,\omega\,u(\hat{x})\qquad\text{on } V.$$

On the other hand, $\mathcal{L}^{\lambda,\omega}(u)(\hat{x})=\lambda\,(u(\hat{x})\langle\omega,1\rangle-\langle\omega,u_{|_V}\rangle)$, which, in particular, implies that

$$-\hat{q}(\hat{x})=\mathcal{L}^{\lambda,\omega}_{\hat{q}}(\hat{\omega})(\hat{x})=\lambda(\langle\omega,1\rangle-n).$$

Therefore, for any $u\in\mathcal{C}(V\cup\{\hat{x}\})$ we get that $\mathcal{L}^{\lambda,\omega}_{\hat{q}}(u)(\hat{x})=\lambda\,(n\,u(\hat{x})-\langle\omega,u_{|_V}\rangle)$, which is equivalent to $\lambda\,\omega\,u(\hat{x})=\dfrac{\omega}{n}\mathcal{L}^{\lambda,\omega}_{\hat{q}}(u)(\hat{x})+\lambda\,\mathcal{P}_\omega(u_{|_V})$ and the second identity for the value of $\mathcal{L}^{\lambda,\omega}_{\hat{q}}(u)$ on V follows. ∎

3 Poisson equations and Green functions

In this section we develop a variational treatment of the discrete analogue of integro-differential equations that allows us to obtain existence and uniqueness results that will be useful to the study of the Green operators associated with the semidefinite positive Schrödinger operators.

Fixed the network Γ, for any ground state $q \in \mathcal{C}(V)$ we consider the following problem, known as the *Poisson equation for $\mathcal{L}_q$ on Γ*:

$$\text{Given } f \in \mathcal{C}(V) \text{ find } u \in \mathcal{C}(V) \text{ such that } \mathcal{L}_q(u) = f. \tag{2}$$

It is well-known that if the ground state is nonnegative and non null, then the Poisson equation has a unique solution for any data $f \in \mathcal{C}(V)$, whereas when $q = 0$ the Poisson equation has a solution for data $f \in \mathcal{C}(V)$ iff $\langle f, 1\rangle = 0$ and moreover the solution is unique up to an additive constant. More generally, when the Schrödinger operator $\mathcal{L}_q$ is positive semidefinite we can tackle the Poisson equation from the following standard variational point of view, see [4, 5].

Proposition 3.1 *Given $\lambda \geqslant 0$, $\omega \in \Omega(V)$, $q = q_\omega + \lambda$ and $f \in \mathcal{C}(V)$ consider the quadratic functional $\mathfrak{J}\colon \mathcal{C}(V) \longrightarrow \mathbb{R}$ determined by*

$$\mathfrak{J}(u) = 2\langle f, u\rangle - \langle \mathcal{L}_q(u), u\rangle, \qquad \textit{for any } u \in \mathcal{C}(V).$$

Then, $u \in \mathcal{C}(V)$ maximizes $\mathfrak{J}$ iff it is a solution of the Poisson equation $\mathcal{L}_q(u) = f$, which implies that $\mathfrak{J}(u) = \langle f, u\rangle = \langle \mathcal{L}_q(u), u\rangle$. Moreover if $\lambda > 0$, then $\mathfrak{J}$ has a unique maximum, whereas when $\lambda = 0$, $\mathfrak{J}$ has a maximum iff $\mathcal{P}_\omega(f) = 0$ in which case there exists a unique maximum up to a multiple of ω.

Corollary 3.2 *Given $\lambda \geqslant 0$, $\omega \in \Omega(V)$, $q = q_\omega + \lambda$ and $f \in \mathcal{C}(V)$, then the Poisson equation $\mathcal{L}_q(u) = f - \mathcal{P}_\omega(f)$ has a unique solution $\hat{u} \in \mathcal{C}(V)$ such that $\mathcal{P}_\omega(\hat{u}) = 0$. Moreover, when $\lambda > 0$, $u = \hat{u} + \lambda^{-1}\mathcal{P}_\omega(f)$ is the unique solution of the equation $\mathcal{L}_q(u) = f$.*

Proof. If we consider $g = f - \mathcal{P}_\omega(f)$ then $\mathcal{P}_\omega(g) = 0$. Therefore, when $\lambda = 0$, from Proposition 3.1 we know that the Poisson equation $\mathcal{L}_q(u) = g$ has solution and that if $v \in \mathcal{C}(V)$ is a solution then the set $\{v + a\omega : a \in \mathbb{R}\}$ describes all the solutions. Moreover, if $\hat{u} = v + a\omega$, then $\mathcal{P}_\omega(\hat{u}) = 0$ iff $a = -\frac{1}{n}\langle \omega, v\rangle$ and hence iff $\hat{u} = v - \mathcal{P}_\omega(v)$. On the other hand when $\lambda > 0$, the Poisson equation $\mathcal{L}_q(u) = g$ has a unique solution and taking into account that $\mathcal{P}_w \circ \mathcal{L}_q = \lambda \mathcal{P}_\omega$ we conclude that $0 = \mathcal{P}_\omega(g) = \lambda \mathcal{P}_\omega(u)$ and hence that $\mathcal{P}_\omega(u) = 0$. Moreover, in this case, we also know that the Poisson equation $\mathcal{L}_q(u) = f$ has a unique solution, say u^*. If we consider $u = \hat{u} + \lambda^{-1}\mathcal{P}_\omega(f)$, then using that $\mathcal{L}_p \circ \mathcal{P}_\omega = \lambda \mathcal{P}_\omega$ we conclude that $\mathcal{L}_q(u) = \mathcal{L}_q(\hat{u}) + \mathcal{P}_\omega(f) = f$, and hence that $u^* = u$. ∎

Given $\omega \in \Omega(V)$ and $\lambda > 0$, Proposition 2.5 establishes that $\mathcal{L}_q(\omega) = \lambda\omega = \lambda \mathcal{P}_\omega(\omega)$ and prompted us to call *generalized Poisson equation on Γ with respect to λ and ω* the following problem, that represents a discrete version of an integro-differential equation:

$$\text{Given } f \in \mathcal{C}(V) \text{ find } u \in \mathcal{C}(V) \text{ such that } \mathcal{L}_q(u) - \lambda \mathcal{P}_\omega(u) = f. \tag{3}$$

As $\mathcal{P}_\omega \circ (\mathcal{L}_q - \lambda \mathcal{P}_\omega) = 0$ the identity $\mathcal{P}_\omega(f) = 0$ is a necessary condition for that the generalized Poisson equation has a solution. Moreover it is also a sufficient condition, since from Corollary 3.2 when $\mathcal{P}_\omega(f) = 0$ there exists a unique solution of the Poisson equation $\mathcal{L}_q(u) = f$ such that $\mathcal{P}_\omega(u) = 0$, which implies that it is also a solution of the generalized Poisson equation. In addition the solution is unique up to a multiple of ω, since from Proposition 2.5 we also know that any solution of the homogeneous generalized Poisson equation must be a multiple of ω. On the other hand, for any $z \in V$ we can solve the so-called *Dirichlet Problem on* $V \setminus \{z\}$.

Proposition 3.3 *Given $\lambda \geqslant 0$, $\omega \in \Omega(V)$, $q = q_\omega + \lambda$ and $f \in \mathcal{C}(V)$, for any $z \in V$ the equation $\mathcal{L}_q(u) - \lambda \mathcal{P}_\omega(u) = f$ on $V \setminus \{z\}$ has a unique solution $u_z \in \mathcal{C}(V)$ such that $u_z(z) = 0$.*

Proof. A function $u \in \mathcal{C}(V)$ satisfies the equation $\mathcal{L}_q(u) - \lambda \mathcal{P}_\omega(u) = f$ on $V \setminus \{z\}$ iff it solves the generalized Poisson equation $\mathcal{L}_q(u) - \lambda \mathcal{P}_\omega(u) = f - \frac{\langle f, \omega \rangle}{w(z)} \varepsilon_z$. Moreover $\{u + a\omega : a \in \mathbb{R}\}$ describes the set of solutions of the generalized Poisson equation and hence, if $v = u + a\omega$, then $v(z) = 0$ iff $a = -\frac{u(z)}{\omega(z)}$. ∎

The above proposition together with Proposition 2.8 allows us to interpret the solution of each non-singular Poisson equation on Γ as the solution of a singular Poisson equation in the host network $\Gamma_{\lambda,\omega}$.

Corollary 3.4 *Given $\lambda > 0$, $\omega \in \Omega(V)$, $q = q_\omega + \lambda$ and $f \in \mathcal{C}(V)$, let $\hat{f} \in \mathcal{C}(V \cup \{\hat{x}\})$ defined as $\hat{f}(x) = f(x)$ for $x \in V$ and $\hat{f}(\hat{x}) = -\langle \omega, f \rangle$. If $u \in \mathcal{C}(V \cup \{\hat{x}\})$ is the unique solution of the Poisson equation $\mathcal{L}_{\hat{q}}^{\lambda,\omega}(u) = \hat{f}$ on $V \cup \{\hat{x}\}$ such that $u(\hat{x}) = 0$, then $u_{|_V}$ is the unique solution of the equation $\mathcal{L}_q(v) = f$ on V.*

Proof. As $\mathcal{P}_{\hat{\omega}}(\hat{f}) = 0$, applying Proposition 3.3 to the Schrödinger operator $\mathcal{L}_{\hat{q}}^{\lambda,\omega}$ we obtain that the Poisson equation $\mathcal{L}_{\hat{q}}^{\lambda,\omega}(u) = \hat{f}$ on $V \cup \{\hat{x}\}$ has a unique solution $u \in \mathcal{C}(V \cup \{\hat{x}\})$ such that $u(\hat{x}) = 0$. Therefore using the identies of Proposition 2.8 we obtain that

$$f = \mathcal{L}_{\hat{q}}^{\lambda,\omega}(u_{|_V}) = \mathcal{L}_q(u_{|_V}) - \lambda\, \omega\, u(\hat{x}) = \mathcal{L}_q(u_{|_V}) \text{ on } V. \quad ∎$$

Definition 3.5 *Consider $\lambda \geqslant 0$, $\omega \in \Omega(V)$ and $q = q_\omega + \lambda$. We call Green operator for Γ, with respect to λ and ω, the endomorphism of $\mathcal{C}(V)$, $\mathcal{G}_{\lambda,\omega}$, that assigns to any $f \in \mathcal{C}(V)$ the unique solution of the Poisson equation $\mathcal{L}_q(u) = f - \mathcal{P}_\omega(f)$ such that $\mathcal{P}_\omega(u) = 0$. The kernel of $\mathcal{G}_{\lambda,\omega}$ is called Green function for Γ, with respect to λ and ω, and it is denoted by $G_{\lambda,\omega}$.*

For any $z \in V$ we call Green operator for $V \setminus \{z\}$, with respect to λ and ω, the endomorphism of $\mathcal{C}(V)$, $\mathcal{G}^z_{\lambda,\omega}$, that assigns to any $f \in \mathcal{C}(V)$ the unique solution of the equation $\mathcal{L}_q(u) - \lambda\mathcal{P}_\omega(u) = f$ on $V \setminus \{z\}$ that vanished at z. The kernel of $\mathcal{G}^z_{\lambda,\omega}$ is called Green function for $V \setminus \{z\}$, with respect to λ and ω, and it is denoted by $G^z_{\lambda,\omega}$.

Definition 3.6 *Consider $\lambda > 0$, $\omega \in \Omega(V)$, $q = q_\omega + \lambda$. We denote by $\mathcal{G}_q$ the inverse of $\mathcal{L}_q$ and by G_q its corresponding kernel.*

As the following result shows the Green operators defined above, or equivalently the Green functions, are closely related. In addition, Corollary 3.4 leads us to obtain the relation between the inverse of positive definite Schrödinger operators and the Green operator for the host network. Moreover, we also show some of the fundamental properties of the kernel associated with all these operators.

Theorem 3.7 *Given $\lambda \geqslant 0$, $\omega \in \Omega(V)$, $q = q_\omega + \lambda$ and $z \in V$, the following properties hold:*

i) $\mathcal{G}_{\lambda,\omega}$ is a self-adjoint and positive semidefinite operator verifying $\mathcal{G}_{\lambda,\omega} \circ \mathcal{P}_\omega = \mathcal{P}_\omega \circ \mathcal{G}_{\lambda,\omega} = 0$, $\mathcal{L}_q \circ \mathcal{G}_{\lambda,\omega} = \mathcal{G}_{\lambda,\omega} \circ \mathcal{L}_q = \mathcal{I} - \mathcal{P}_\omega$ and hence $(\mathcal{L}_q - \lambda\mathcal{P}_\omega) \circ \mathcal{G}_{\lambda,\omega} \circ (\mathcal{L}_q - \lambda\mathcal{P}_\omega) = (\mathcal{L}_q - \lambda\mathcal{P}_\omega)$. In addition, $G_{\lambda,\omega}(x,x) > 0$ for any $x \in V$ and moreover when $\lambda > 0$, then

$$-\frac{1}{n\lambda}\,\omega(x)\omega(y) < G_{\lambda,\omega}(x,y) < G_{\lambda,\omega}(y,y)\frac{\omega(x)}{\omega(y)} \quad \text{for any } x, y \in V \text{ with } x \neq y.$$

ii) $\mathcal{G}^z_{\lambda,\omega}$ is a self-adjoint and positive definite operator and $\mathcal{G}^z_{\lambda,\omega} = \mathcal{G}_{\lambda,\omega} + a\mathcal{P}_\omega + \mathcal{P}_{\omega,\tau} + \mathcal{P}_{\tau,\omega}$, where $a = \dfrac{nG_{\lambda,\omega}(z,z)}{\omega^2(z)}$ and $\tau = -\dfrac{G_{\lambda,\omega}(\cdot,z)}{\omega(z)}$. Moreover, $G^z_{\lambda,\omega}(\cdot,z) = 0$ and for any $x, y \in V \setminus \{z\}$ it is verified that $0 \leqslant G^z_{\lambda,\omega}(x,y)\omega(y) \leqslant G^z_{\lambda,\omega}(y,y)\omega(x)$, where the first inequality is an equality iff $\lambda = 0$ and z separates x and y, whereas the second one is an equality iff $\lambda = 0$ and y separates x and z.

iii) If $\lambda > 0$, then $\mathcal{G}_q$ is a self-adjoint and positive definite operator verifying $\mathcal{G}_q = \mathcal{G}_{\lambda,\omega} + \lambda^{-1}\mathcal{P}_\omega$. Moreover, if $\widehat{G}$ and $\widehat{G}^{\hat{x}}$ are the Green functions for $\Gamma_{\lambda,\omega}$ and for V respectively, then $\widehat{G}(\cdot,\hat{x}) = \dfrac{(n+1)\,\varepsilon_{\hat{x}} - \widehat{\omega}}{\lambda(n+1)^2}$, $G_q = \widehat{G}^{\hat{x}}_{|V\times V}$ and $G_{\lambda,\omega} = \widehat{G}_{|V\times V} - \dfrac{\omega \otimes \omega}{\lambda\, n(n+1)^2}$.

Proof. As $\mathcal{L}_q$ and $\mathcal{L}_q - \lambda\mathcal{P}_\omega$ are self-adjoint operators, by using standard techniques, we can deduce that the operators $\mathcal{G}_{\lambda,\omega}$, $\mathcal{G}^z_{\lambda,\omega}$ and $\mathcal{G}_q$ are also self-adjoint operators. The same occurs with the positive-semidefiniteness and the positive definiteness.

(i) The proof of the properties related with composition is straightforward. On the other hand, given $y \in V$, then $u = G_{\lambda,\omega}(\cdot,y) = \mathcal{G}_{\lambda,\omega}(\varepsilon_y)$ and hence $\mathcal{L}_q(u) = \varepsilon_y - \frac{1}{n}\omega\omega(y) \neq 0$. Applying the positive semidefiniteness of $\mathcal{L}_q$ we obtain that $0 < \langle \mathcal{L}_q(u), u\rangle = u(y) - \mathcal{P}_\omega(u) = u(y)$ and hence $G_{\lambda,\omega}(y,y)>0$. On the other hand, if $\lambda>0$, taking $v=u+\dfrac{\omega(y)}{n\,\lambda}\,\omega$ and $h=\dfrac{\omega}{\omega(y)}-\dfrac{u}{u(y)}$, then $v(y)>0$, $h(y)=0$, $\mathcal{L}_q(v)=\varepsilon_y\geqslant 0$ on V and $\mathcal{L}_q(h)=\left(\dfrac{\lambda}{\omega(y)}+\dfrac{\omega(y)}{n\,u(y)}\right)\omega-\dfrac{\varepsilon_y}{u(y)}>0$ on $V\setminus\{y\}$. Therefore, by applying the well-known *Minimum Principle*, see for instance [4, Proposition 4.16], we obtain that $v,h>0$ on $V\setminus\{y\}$ and the two claimed inequalities follow.

(ii) It is clear that $\mathcal{G}^z_{\lambda,\omega}(\varepsilon_z)=0$; that is, $G^z_{\lambda,\omega}(\cdot,z)=0$. Moreover, given $y\in V$, then from Proposition 3.3 the function $u=G^z_{\lambda,\omega}(\cdot,y)=\mathcal{G}^z_{\lambda,\omega}(\varepsilon_y)$ is given by

$$
\begin{aligned}
u&=\mathcal{G}_{\lambda,\omega}(\varepsilon_y)-\frac{\omega(y)}{\omega(z)}\,\mathcal{G}_{\lambda,\omega}(\varepsilon_z)-\frac{\mathcal{G}_{\lambda,\omega}(\varepsilon_y)(z)}{\omega(z)}\,\omega+\frac{\omega(y)}{\omega^2(z)}\,\mathcal{G}_{\lambda,\omega}(\varepsilon_z)(z)\,\omega\\
&=\mathcal{G}_{\lambda,\omega}(\varepsilon_y)+\tau\,\omega(y)+\omega\,\tau(y)+a\omega\,\omega(y)
\end{aligned}
$$

which implies that $\mathcal{G}^z_{\lambda,\omega}=\mathcal{G}_{\lambda,\omega}+a\mathcal{P}_\omega+\mathcal{P}_{\omega,\tau}+\mathcal{P}_{\tau,\omega}$. As $\langle\omega,\tau\rangle=0$, from this identity we obtain that $\mathcal{G}^z_{\lambda,\omega}(\omega)=a\omega+n\tau$ and hence that

$$
\mathcal{P}_\omega(u)=\frac{\omega}{n}\,\langle\omega,u\rangle=\frac{\omega}{n}\,\langle\mathcal{G}^z_{\lambda,\omega}(\omega),\varepsilon_y\rangle=\frac{\omega}{n}\Big(a\omega(y)+n\tau(y)\Big)=\frac{\omega}{\omega(z)}\Big(G_{\lambda,\omega}(z,z)\,\frac{\omega(y)}{\omega(z)}-G_{\lambda,\omega}(y,z)\Big),
$$

which, after part (i), implies that $\mathcal{P}_\omega(u)\geqslant 0$. On the other hand, $u(z)=0$ and moreover $\mathcal{L}_q(u)=\varepsilon_y+\lambda\mathcal{P}_\omega(u)\geqslant 0$ on $V\setminus\{z\}$. Therefore applying newly the Minimum Principle, we obtain that $u\geqslant 0$ on V and if V_{yz} is the connected component of $V\setminus\{z\}$ that contains y, then $u(x)>0$ for any $x\in V_{yz}$, since $\mathcal{L}_q(u)(y)=1+\lambda\mathcal{P}_\omega(u)(y)>0$. This implies that if $0=u(y)=G^z_{\lambda,\omega}(x,y)$, necessarily $\lambda=0$ and $x\notin V_{yz}$; *i.e.*, z separates x and y.

Finally if we consider $v=G^y_{\lambda,\omega}(\cdot,z)$ and $h=\omega(z)u+\omega(y)v$, then $\mathcal{L}_q(h)-\lambda\mathcal{P}_\omega(h)=0$ and hence there exists $\alpha\in\mathbb{R}$ such that $h=\alpha\omega$. This equality implies that $a\omega(y)=h(y)=\omega(z)u(y)=G^z_{\lambda,\omega}(y,y)\omega(z)$ and hence that

$$
G^z_{\lambda,\omega}(x,y)\leqslant G^z_{\lambda,\omega}(x,y)+G^y_{\lambda,\omega}(x,z)\frac{\omega(z)}{\omega(y)}=G^z_{\lambda,\omega}(y,y)\,\frac{\omega(z)}{\omega(y)},
$$

since $G^y_{\lambda,\omega}(x,z)\geqslant 0$. Moreover the inequality is an equality iff $G^y_{\lambda,\omega}(x,z)=0$; that is, iff $\lambda=0$ and y separates x and z.

(iii) As $\mathcal{L}_q\circ(\mathcal{G}_{\lambda,\omega}+\lambda^{-1}\mathcal{P}_\omega)=\mathcal{I}-\mathcal{P}_\omega+\lambda^{-1}\lambda\mathcal{P}_\omega=\mathcal{I}$, we conclude that $\mathcal{G}_q=\mathcal{G}_{\lambda,\omega}+\lambda^{-1}\mathcal{P}_\omega$. If $u=\widehat{G}(\cdot,\hat{x})$, then $\widehat{\mathcal{L}}_{\hat{q}}(u)=\varepsilon_{\hat{x}}-\dfrac{\hat{\omega}}{n+1}$, which implies that $\widehat{\mathcal{L}}_{\hat{q}}(u)(\hat{x})=\dfrac{n}{n+1}$ and hence that $\mathcal{L}_q(u_{|_V})-\lambda\,\mathcal{P}_\omega(u_{|_V})=0$ on V. Therefore, $u_{|_V}=a\omega$, $a\in\mathbb{R}$, $u(\hat{x})=\dfrac{1}{\lambda n}\,\widehat{\mathcal{L}}_{\hat{q}}(u)(\hat{x})+\dfrac{1}{n}\langle\omega,u_{|_V}\rangle=\dfrac{1}{\lambda\,(n+1)}+a$ and hence $u=a\widehat{\omega}+\dfrac{1}{\lambda(n+1)}\,\varepsilon_{\hat{x}}$. Finally, the condition $\langle\widehat{\omega},u\rangle=0$ implies that $a=-\dfrac{1}{\lambda(n+1)^2}$ and the first identity follows.

On the other hand, let $y\in V$ and consider $v=\widehat{G}(\cdot,y)$. Then, $\widehat{\mathcal{L}}_{\hat{q}}(v)=\varepsilon_y-\dfrac{\widehat{\omega}\,\omega(y)}{n+1}$, which implies that $\widehat{\mathcal{L}}_{\hat{q}}(v)(\hat{x})=-\dfrac{\omega(y)}{n+1}$ and hence that $\mathcal{L}_q(v_{|_V})-\lambda\,\mathcal{P}_\omega(v_{|_V})=\varepsilon_y-\dfrac{1}{n}\,\omega\,\omega(y)$ on V.

Therefore, $v_{|_V} = G_{\lambda,\omega}(\cdot, y) + a\omega$ where $an = \langle \omega, v_{|_V}\rangle$, since $\langle G_{\lambda,\omega}(\cdot, y), \omega\rangle = 0$. The same reason implies that $0 = \langle \widehat{\omega}, v\rangle = v(\hat{x}) + \langle \omega, v_{|_V}\rangle$ and hence that $\langle \omega, v_{|_V}\rangle = \dfrac{\omega(y)}{\lambda(n+1)^2}$. In conclusion, we obtain that $v_{|_V} = G_{\lambda,\omega}(\cdot, y) + \dfrac{1}{\lambda\, n(n+1)^2}\,\omega\,\omega(y)$. From this equality and applying part (ii), for any $x, y \in V$ we get that

$$\begin{aligned}\widehat{G}^{\hat{x}}(x,y) &= \widehat{G}(x,y) - \widehat{G}(x,\hat{x})\omega(y) - \widehat{G}(y,\hat{x})\omega(x) + \widehat{G}(\hat{x},\hat{x})\,\omega(x)\omega(y)\\ &= \widehat{G}(x,y) + \frac{(n+2)}{\lambda(n+1)^2}\,\omega(x)\omega(y) = G_{\lambda,\omega}(x,y) + \frac{1}{\lambda\, n}\,\omega(x)\,\omega(y) = G_q(x,y). \quad \blacksquare\end{aligned}$$

We finish this section obtaining the matrix counterpart of some of the above results.

Corollary 3.8 *Assume that the vertices of V are labelled from 1 to n and consider $\lambda \geqslant 0$, $\omega \in \Omega(V)$, $q = q_\omega + \lambda$ and $\mathsf{G}_{\lambda,\omega}$ the matrix of order n identified with $G_{\lambda,\omega}$. Let $z = x_n$, $\mathsf{w}^z \in \mathbb{R}^{n-1}$ obtained from w by deleting its n-th component and L^z, G^z, P^z the matrices of order $n-1$ obtained by deleting the n-th row and column of $\mathsf{L}_\omega + \lambda(\mathsf{I} - \mathsf{P}_\omega)$, of the matrix identified with $G^z_{\lambda,\omega}$ and of P_ω, respectively. Then, L^z and G^z are mutually inverses and moreover*

$$[\mathsf{L}_\omega + \lambda(\mathsf{I} - \mathsf{P}_\omega)]^\dagger = \mathsf{G}_{\lambda,\omega} = \frac{(\mathsf{w}^z)^*\mathsf{G}^z\mathsf{w}^z}{n}\,\mathsf{P}_\omega + \begin{bmatrix} \mathsf{G}^z - \mathsf{G}^z\mathsf{P}^z_\omega - \mathsf{P}^z_\omega\mathsf{G}^z & -\frac{1}{n}\mathsf{G}^z\mathsf{w}^z \\ -\frac{1}{n}(\mathsf{w}^z)^*\mathsf{G}^z & 0 \end{bmatrix},$$

where $\dagger$ stands for the Moore-Penrose inverse.

Observe that when $\lambda = 0$ and $\omega = 1$, then $\mathsf{L}_\omega + \lambda(\mathsf{I} - \mathsf{P}_\omega) = \mathsf{L}$ and the above expression becomes

$$\mathsf{L}^\dagger = \frac{1^*\mathsf{G}^z 1}{n^2}\,\mathsf{J} + \begin{bmatrix} \mathsf{G}^z - \frac{1}{n}\mathsf{G}^z\mathsf{J} - \frac{1}{n}\mathsf{J}\mathsf{G}^z & -\frac{1}{n}\mathsf{G}^z 1 \\ -\frac{1}{n}1^*\mathsf{G}^z & 0 \end{bmatrix},$$

where J denotes the matrix all whose entries equal 1. This equality was obtained in [12], where the matrix G^z was called the *bottleneck matrix of* L *based at* z.

4 The effective resistances of a network

In the standard setting, the effective resistance between vertices x and y is defined through the solution of the Poisson equation $\mathcal{L}(u) = f$ when the data is the dipole with poles at x and y; that is, $f = \varepsilon_x - \varepsilon_y$. The knowledge of the effective resistance can be used to deduce important properties of electrical networks, see for instance [10]. One of them establishes that the Green function of the complement of any vertex can be obtained in terms of the effective resistances, see for instance [8, 11]. Moreover, Corollary 3.8 and the relation between M-matrices and Schrödinger operators given in Lemma 2.2 imply that any irreducible and weakly diagonal dominant Stieltjes matrix is the resistive inverse associated with a suitable network Γ, which is precisely the main result in [8].

Throughout this section we generalize the above mentioned facts in several ways. First we use the definition of the dipole with respect to a weight introduced in [4] to define the concept of the effective resistance between two vertices with respect to a value $\lambda \geqslant 0$ and a weight $\omega \in \Omega(V)$. So, we obtain that the Green function of the complement of any vertex in V, associated with a positive semidefinite Schrödinger operator, singular or not, can be expressed in terms of the effective resistances with respect to a non-negative value and a weight. As a by-product, we give a new version of Fiedler's result and moreover we can eliminate the hypothesis of diagonally dominance to obtain that any irreducible Stieltjes matrix is a resistive inverse. On the other hand we introduce here the concept of total resistance of a vertex with respect to a positive value and a weight, that in some sense generalized the notion of *status of a vertex* introduced in [12], and that together with effective resistances allows us to obtain the expression of the Green functions. The matrix version of these results leads to express the Moore-Penrose inverse of any irreducible symmetric M-matrix in terms of the matrix of effective resistance.

In the sequel we consider fixed the network $\Gamma = (V, c)$, the value $\lambda \geqslant 0$, the weight $\omega \in \Omega(V)$ and $\mathcal{L}_q$ the Schrödinger operator with ground state $q = q_\omega + \lambda$. Given $x, y \in V$, the *ω-dipole between x and y* is the function $f_{xy} = \dfrac{1}{\omega}(\varepsilon_x - \varepsilon_y)$. Observe that $f_{xx} = 0$ for any $x \in V$, whereas when $\omega = 1$ the ω-dipole between x and y is simply the standard dipole. Clearly, for any $x, y \in V$ it is verified that $\mathcal{P}_\omega(f_{xy}) = 0$ and then the hypotheses of Proposition 3.1 are in force. Consequently, given $x, y \in V$ the functional $\mathfrak{J}_{x,y}\colon \mathcal{C}(V) \longrightarrow \mathbb{R}$ determined for any $u \in \mathcal{C}(V)$ by the expression

$$\mathfrak{J}_{x,y}(u) = 2\left[\frac{u(x)}{\omega(x)} - \frac{u(y)}{\omega(y)}\right] - \langle \mathcal{L}_q(u), u\rangle \tag{4}$$

attains a maximum value. In addition, $v \in \mathcal{C}(V)$ maximizes $\mathfrak{J}_{x,y}$ iff satisfies the Poisson equation $\mathcal{L}_q(v) = f_{xy}$.

On the other hand, given $x \in V$ we consider the function $f_x = \dfrac{1}{\omega(x)}\Big(\varepsilon_x - \mathcal{P}_\omega(\varepsilon_x)\Big)$, that clearly satisfies $\mathcal{P}_\omega(f_x) = 0$ which implies that the hypotheses of Proposition 3.1 are newly in force. So, the functional $\mathfrak{J}_x\colon \mathcal{C}(V) \longrightarrow \mathbb{R}$ determined for any $u \in \mathcal{C}(V)$ by the expression

$$\mathfrak{J}_x(u) = 2\left[\frac{u(x)}{\omega(x)} - \frac{1}{n}\langle u, \omega\rangle\right] - \langle \mathcal{L}_q(u), u\rangle \tag{5}$$

attains a maximum value. In addition, $v \in \mathcal{C}(V)$ maximizes $\mathfrak{J}_x$ iff satisfies the Poisson equation $\mathcal{L}_q(v) = f_x$.

Definition 4.1 *Given $x, y \in V$, we define the Effective Resistance between x and y and the Total Resistance at $x \in V$, with respect to λ and ω, as the values*

$$R_{\lambda,\omega}(x, y) = \max_{u \in \mathcal{C}(V)} \{\mathfrak{J}_{x,y}(u)\} \quad \text{and} \quad r_{\lambda,\omega}(x) = \max_{u \in \mathcal{C}(V)} \{\mathfrak{J}_x(u)\},$$

respectively. Moreover we call Kirchhoff Index of Γ*, with respect to* λ *and* ω*, the value*

$$\mathsf{k}(\lambda,\omega) = \frac{1}{2}\sum_{x,y\in V} R_{\lambda,\omega}(x,y)\omega^2(x)\,\omega^2(y).$$

The kernel $R_{\lambda,\omega}\colon V \times V \longrightarrow \mathbb{R}$ *is called the Effective Resistance of the network* Γ*, with respect to* λ *and* ω*, whereas its associated endomorphism,* $\mathcal{R}_{\lambda,\omega}$*, is called Effective Resistance operator of the network* Γ*, with respect to* λ *and* ω*. The function* $r_{\lambda,\omega}\colon V \longrightarrow \mathbb{R}$ *is called the Total Resistance of the network* Γ*, with respect to* λ *and* ω*.*

Observe that $R_{\lambda,\omega}(x,x) = 0$ for any $x \in V$, since the functional $\mathfrak{J}_{x,x}$ attains its maximum value at $v = a\omega$, where $a = 0$ when $\lambda > 0$ and $a \in \mathbb{R}$, otherwise.

In the sequel we omit the expression with respect to λ and ω when it does not lead to confusion. When $\lambda = 0$ we usually omit the subindex λ in the above expressions and when, in addition, $\omega = 1$ we also omit the subindex ω. Therefore, R is nothing else than the standard effective resistance of the network, whereas k is the Kirchhoff Index introduced in the context of Organic Chemistry, see for instance [19] and which study is the main theme in [10].

Proposition 4.2 *If* $u, v \in \mathcal{C}(V)$ *are solutions of the Poisson equations* $\mathcal{L}_q(u) = f_{xy}$ *and* $\mathcal{L}_q(v) = f_x$*, respectively, then*

$$R_{\lambda,\omega}(x,y) = \langle \mathcal{L}_q(u), u\rangle = \frac{u(x)}{\omega(x)} - \frac{u(y)}{\omega(y)} \quad \text{and} \quad r_{\lambda,\omega}(x) = \langle \mathcal{L}_q(v), v\rangle = \frac{v(x)}{\omega(x)} - \frac{1}{n}\,\langle v,\omega\rangle.$$

Therefore, $r_{\lambda,\omega}$ *is positive, whereas* $R_{\lambda,\omega}$ *is symmetric, non-negative and moreover* $R_{\lambda,\omega}(x,y) = 0$ *iff* $x = y$*.*

Proof. The first claims are straightforward consequences of Proposition 3.1. On the other hand, given $u \in \mathcal{C}(V)$ we get that $\mathfrak{J}_{x,y}(u) = \mathfrak{J}_{y,x}(-u)$ and hence $R_{\lambda,\omega}(x,y) = R_{\lambda,\omega}(y,x)$ for any $x, y \in V$. Moreover, we know that $R_{\lambda,\omega}(x,x) = 0$ for any $x \in V$ and also that $R_{\lambda,\omega}(x,y) = 0$ iff $\langle \mathcal{L}_q(u), u\rangle = 0$ for any solution of the Poisson equation $\mathcal{L}_q(u) = f_{xy}$. So, $u = a\omega$, where $a = 0$ if $\lambda > 0$, that in any case implies that $\mathcal{L}_q(u) = 0$ and hence $f_{xy} = 0$ or equivalently $x = y$. The proof for the positiveness of $r_{\lambda,\omega}$ follows by the same arguments. ∎

The relation between the effective resistance, the total resistance and Poisson equations, leads us to the following relations between the effective resistances and the Green operator for Γ that is nothing else than a generalization of the well-known characterization of the so-called *Campbell-Youla inverse*, see [18]. Moreover we obtain the characterization of the effective resistance in terms of the trace of the matrix identified with the Green operator, a well-known property in the standard setting, see for instance [10, 11]. In addition, we show that the effective resistance is a non singular operator and we give the formula for its inverse, that in the standard case can be found in [2].

Proposition 4.3 *For any $x \in V$ we get that $r_{\lambda,\omega}(x) = \dfrac{G_{\lambda,\omega}(x,x)}{\omega^2(x)}$. Moreover,*

$$\mathcal{R}_{\lambda,\omega} = \mathcal{P}_{r_{\lambda,\omega},1} + \mathcal{P}_{1,r_{\lambda,\omega}} - 2\mathcal{D}_\omega^{-1} \circ \mathcal{G}_{\lambda,\omega} \circ \mathcal{D}_\omega^{-1}$$

$$\mathcal{G}_{\lambda,\omega} = -\frac{1}{2}\mathcal{D}_\omega \circ \Big(\mathcal{I} - \frac{1}{n}\mathcal{P}_{1,\omega^2}\Big) \circ \mathcal{R}_{\lambda,\omega} \circ \Big(\mathcal{I} - \frac{1}{n}\mathcal{P}_{\omega^2,1}\Big) \circ \mathcal{D}_\omega$$

and hence $\frac{1}{n}\,\mathsf{k}(\lambda,\omega) = \sum\limits_{x\in V} G_{\lambda,\omega}(x,x) = \langle r_{\lambda,\omega}, \omega^2\rangle = \mathcal{R}_{\lambda,\omega}(\omega^2) - n\, r_{\lambda,\omega}$. In addition, $\mathcal{R}_{\lambda,\omega}$ is invertible and if we take $\nu = \omega(\mathcal{L}_q - \lambda\mathcal{P}_\omega)(\omega r_{\lambda,\omega}) + \dfrac{2}{n}\,\omega^2$, then

$$\mathcal{R}_{\lambda,\omega}^{-1} = -\frac{1}{2}\,\mathcal{D}_\omega \circ (\mathcal{L}_q - \lambda\mathcal{P}_\omega) \circ \mathcal{D}_\omega + \frac{\langle \nu,\nu\rangle}{\langle \mathcal{R}_{\lambda,\omega}(\nu),\nu\rangle}\,\mathcal{P}_\nu.$$

Proof. If $v = \mathcal{G}_{\lambda,\omega}(f_x)$, then $\mathcal{L}_q(v) = f_x$ and hence $r_{\lambda,\omega}(x) = \dfrac{v(x)}{\omega(x)} - \dfrac{1}{n}\,\langle v,\omega\rangle = \dfrac{v(x)}{\omega(x)}$, since $\mathcal{P}_\omega(v) = 0$. The first claim follows taking into account that $v = \dfrac{1}{\omega}\,\mathcal{G}_{\lambda,\omega}(\varepsilon_x) = \dfrac{1}{\omega(x)}\,G_{\lambda,\omega}(\cdot,x)$, since $\mathcal{G}_{\lambda,\omega} \circ \mathcal{P}_\omega = 0$ and the obtained expression for $r_{\lambda,\omega}$ implies that $\sum\limits_{x\in V} G_{\lambda,\omega}(x,x) = \langle r_{\lambda,\omega}, \omega^2\rangle$.

On the other hand, given $x,y \in \mathbb{R}$ and $u = \mathcal{G}_{\lambda,\omega}(f_{xy})$ then $\mathcal{L}_q(u) = f_{xy}$ and hence $R_{\lambda,\omega}(x,y) = \dfrac{u(x)}{\omega(x)} - \dfrac{u(y)}{\omega(y)}$. As $u = \dfrac{\mathcal{G}_{\lambda,\omega}(\varepsilon_x)}{\omega(x)} - \dfrac{\mathcal{G}_{\lambda,\omega}(\varepsilon_y)}{\omega(y)} = \dfrac{G_{\lambda,\omega}(\cdot,\varepsilon_x)}{\omega(x)} - \dfrac{G_{\lambda,\omega}(\cdot,\varepsilon_y)}{\omega(y)}$, we get

$$R_{\lambda,\omega}(x,y) = r_{\lambda,\omega}(x) + r_{\lambda,\omega}(y) - \frac{2G_{\lambda,\omega}(x,y)}{\omega(x)\omega(y)},$$

where we have used that $G_{\lambda,\omega}$ is symmetric. From this equality we have that

$$\mathcal{R}_{\lambda,\omega} = \mathcal{P}_{r_{\lambda,\omega},1} + \mathcal{P}_{1,r_{\lambda,\omega}} - 2\mathcal{D}_\omega^{-1} \circ \mathcal{G}_{\lambda,\omega} \circ \mathcal{D}_\omega^{-1}$$

which implies that $\mathcal{R}_{\lambda,\omega}(\omega^2) = n\, r_{\lambda,\omega} + \langle r_{\lambda,\omega}, \omega^2\rangle - 2\mathcal{D}_\omega^{-1}(\mathcal{G}_{\lambda,\omega}(\omega)) = n\, r_{\lambda,\omega} + \dfrac{1}{n}\,k(\lambda,\omega)$, where we have take newly into account that $\mathcal{G}_{\lambda,\omega}(\omega) = 0$. Now this identity implies that

$$\mathsf{k}(\lambda,\omega) = \frac{1}{2}\,\langle \mathcal{R}_{\lambda,\omega}(\omega^2), \omega^2\rangle = n\,\langle r_{\lambda,\omega}, \omega^2\rangle + k(\lambda,\omega),$$

and hence that $\mathsf{k}(\lambda,\omega) = n \sum\limits_{x\in V} G_{\lambda,\omega}(x,x)$. In addition, from the obtained expression for $\mathcal{R}_{\lambda,\omega}$ we get that $\mathcal{G}_{\lambda,\omega} = \frac{1}{2}\, D_\omega \circ \big[\mathcal{P}_{r_{\lambda,\omega},1} + \mathcal{P}_{1,r_{\lambda,\omega}} - \mathcal{R}_{\lambda,\omega}\big]\,\mathcal{D}_\omega$ which in turn is equivalent to the identity $\mathcal{G}_{\lambda,\omega} = -\dfrac{1}{2}\mathcal{D}_\omega \circ \Big(\mathcal{I} - \dfrac{1}{n}\mathcal{P}_{1,\omega^2}\Big) \circ \mathcal{R}_{\lambda,\omega} \circ \Big(\mathcal{I} - \dfrac{1}{n}\mathcal{P}_{\omega^2,1}\Big) \circ \mathcal{D}_\omega$.

If we consider the operator $\mathcal{K} = -\frac{1}{2}\mathcal{D}_\omega \circ \mathcal{R}_{\lambda,\omega} \circ \mathcal{D}_\omega$ and its kernel K, then $\mathcal{K}$ is self-adjoint and $K(x,x) = 0$ for any $x \in V$, since $\mathcal{R}_{\lambda,\omega}$ and $R_{\lambda,\omega}$ satisfy the same properties. In addition, from the obtained expression for $\mathcal{R}_{\lambda,\omega}$, we get that $\mathcal{K} = \mathcal{G}_{\lambda,\omega} + \mathcal{P}_{\omega,\tau} + \mathcal{P}_{\tau,\omega}$, where $\tau = -\frac{1}{2}\,\omega\, r_{\lambda,\omega}$

and hence that $\langle \mathcal{K}(u), u\rangle > 0$ for any $u \in \mathcal{C}(V)$ such that $\mathcal{P}_\omega(u) = 0$. So, any matrix identified with $\mathcal{K}$ is a zero-axial and conditionally positive definite, with respect to ω, matrix and hence an invertible matrix by applying [3, Lemma 4.3.5]. This implies that $\mathcal{R}_{\lambda,\omega}$ is a non singular operator. Moreover, if $\nu = \omega(\mathcal{L}_q - \lambda\mathcal{P}_\omega)(\omega r_{\lambda,\omega}) + \dfrac{2}{n}\,\omega^2$, then

$$-\frac{1}{2}\,\mathcal{D}_\omega \circ (\mathcal{L}_q - \lambda\mathcal{P}_\omega) \circ \mathcal{D}_\omega \circ \mathcal{R}_{\lambda,\omega} = -\frac{1}{2}\mathcal{P}_{\nu,1} + \frac{1}{n}\mathcal{P}_{\omega^2,1} + \mathcal{D}_\omega \circ (\mathcal{I} - \mathcal{P}_\omega)\mathcal{D}_\omega^{-1} = -\frac{1}{2}\mathcal{P}_{\nu,1} + \mathcal{I}$$

since $(\mathcal{L}_q - \lambda\mathcal{P}_\omega) \circ \mathcal{G}_{\lambda,\omega} = \mathcal{I} - \mathcal{P}_\omega$ and $(\mathcal{L}_q - \lambda\mathcal{P}_\omega)(\omega) = 0$. On the other hand,

$$\langle \nu, 1\rangle = \langle (\mathcal{L}_q - \lambda\mathcal{P}_\omega)(\omega r_{\lambda,\omega}), \omega\rangle + \frac{2}{n}\langle \omega, \omega\rangle = \langle \omega r_{\lambda,\omega}, (\mathcal{L}_q - \lambda\mathcal{P}_\omega)(\omega)\rangle + 2 = 2.$$

Therefore, $\mathcal{P}_{\nu,1}(\nu) = 2\nu$ and hence $-\frac{1}{2}\,\mathcal{D}_\omega \circ (\mathcal{L}_q - \lambda\mathcal{P}_\omega) \circ \mathcal{D}_\omega \circ \mathcal{R}_{\lambda,\omega}(\nu) = -\frac{1}{2}\mathcal{P}_{\nu,1}(\nu) + \nu = 0$. This equality implies that $\omega\mathcal{R}_{\lambda,\omega}(\nu) = a\omega$, that is, $\mathcal{R}_{\lambda,\omega}(\nu) = a \cdot 1$, where $a \neq 0$ since $\nu \neq 0$ and $\mathcal{R}_{\lambda,\omega}$ is non-singular. Moreover, $a = \frac{1}{2}\,\langle \mathcal{R}_{\lambda,\omega}(\nu), \nu\rangle$ and

$$-\frac{1}{2}\mathcal{P}_{\nu,1} = -\frac{1}{\langle \mathcal{R}_{\lambda,\omega}(\nu), \nu\rangle}\,\mathcal{P}_{\nu,\mathcal{R}_{\lambda,\omega}(\nu)} = -\frac{\langle \nu, \nu\rangle}{\langle \mathcal{R}_{\lambda,\omega}(\nu), \nu\rangle}\,\mathcal{P}_\nu \circ \mathcal{R}_{\lambda,\omega}.$$

Finally, we obtain that

$$-\frac{1}{2}\,\mathcal{D}_\omega \circ (\mathcal{L}_q - \lambda\mathcal{P}_\omega) \circ \mathcal{D}_\omega \circ \mathcal{R}_{\lambda,\omega} = -\frac{\langle \nu, \nu\rangle}{\langle \mathcal{R}_{\lambda,\omega}(\nu), \nu\rangle}\,\mathcal{P}_\nu \circ \mathcal{R}_{\lambda,\omega} + \mathcal{I}$$

and the last claim follows. ∎

Our next aim is to re-write the above identities in terms of the kernels associated with the involved operators. In addition, by applying part (ii) of Proposition 3.7, we can also obtain a generalization of the well-known expression of the Green function for the complement of a vertex in terms of the effective resistances. As a by-product, we prove that the generalized effective resistance is a distance, see for instance [19] for the standard case and [4, 5] for the case $\lambda = 0$.

Corollary 4.4 *For any $x, y, z \in V$ the following identities hold*

$$G_{\lambda,\omega}(x,y) = \frac{1}{2n}\,\omega(x)\omega(y)\sum_{t\in V}\Big(R_{\lambda,\omega}(x,t) + R_{\lambda,\omega}(y,t) - R_{\lambda,\omega}(x,y)\Big)\omega^2(t) - \frac{1}{n^2}\,\omega(x)\omega(y)\,\mathsf{k}(\lambda,\omega),$$

$$G^z_{\lambda,\omega}(x,y) = \frac{1}{2}\,\omega(x)\omega(y)\Big(R_{\lambda,\omega}(x,z) + R_{\lambda,\omega}(y,z) - R_{\lambda,\omega}(x,y)\Big).$$

In particular, $R_{\lambda,\omega}$ defines a distance on Γ and $R_{\lambda,\omega}(x,y) = R_{\lambda,\omega}(x,z) + R_{\lambda,\omega}(y,z)$ iff $\lambda = 0$ and z separates x and y. In addition, $\Big|r_{\lambda,\omega}(x) - r_{\lambda,\omega}(y)\Big| \leqslant R_{\lambda,\omega}(x,y)$ for any $x, y \in V$, with equality iff $x = y$ and when $\lambda > 0$, it is also verified $R_{\lambda,\omega}(x,y) < r_{\lambda,\omega}(x) + r_{\lambda,\omega}(y) + \frac{2}{n\lambda}$.

Proof. From the above proposition we get that $r_{\lambda,\omega}(x) = \dfrac{1}{n}\sum\limits_{t\in V} R_{\lambda,\omega}(x,t)\omega^2(t) - \dfrac{1}{n^2}\,\mathsf{k}(\lambda,\omega)$ and that

$$G_{\lambda,\omega}(x,y) = \frac{1}{2}\,\omega(x)\omega(y)\Big(r_{\lambda,\omega}(x) + r_{\lambda,\omega}(y) - R_{\lambda,\omega}(x,y)\Big)$$

which, taking into account that $\sum\limits_{t\in V} R_{\lambda,\omega}(x,y)\omega^2(t) = nR_{\lambda,\omega}(x,y)$, implies the first identity.

On the other hand, from the identity $\mathcal{G}^z_{\lambda,\omega} = \mathcal{G}_{\lambda,\omega} + a\mathcal{P}_\omega + \mathcal{P}_{\omega,\tau} + \mathcal{P}_{\tau,\omega}$, we deduce that

$$R_{\lambda,\omega}(x,y) = \frac{G_{\lambda,\omega}(x,x)}{\omega^2(x)} + \frac{G_{\lambda,\omega}(y,y)}{\omega^2(y)} - \frac{2G_{\lambda,\omega}(x,y)}{\omega(x)\omega(y)} = \frac{G^z_{\lambda,\omega}(x,x)}{\omega^2(x)} + \frac{G^z_{\lambda,\omega}(y,y)}{\omega^2(y)} - \frac{2G^z_{\lambda,\omega}(x,y)}{\omega(x)\omega(y)}.$$

In particular, $G^z_{\lambda,\omega}(x,x) = R_{\lambda,\omega}(x,z)\omega^2(x)$, for any $x \in V$, since $G^z_{\lambda,\omega}(x,z) = 0$ and the second identity follows. The rest of the results follow straightforwardly from the inequalities in Proposition 3.7. ∎

Corollary 4.5 *If $\mathsf{R}_{\lambda,\omega}$ and S_ω are the matrices identified with $\mathcal{R}_{\lambda,\omega}$, and $\frac{1}{n}\mathcal{P}_{\omega^2,1}$ respectively, then it is verified that*

$$\Big(\mathsf{L}_\omega + \lambda(\mathsf{I}-\mathsf{P}_\omega)\Big)^\dagger = -\frac{1}{2}\,\mathsf{D}_\omega(\mathsf{I}-\mathsf{S}^*_\omega)\mathsf{R}_{\lambda,\omega}(\mathsf{I}-\mathsf{S}_\omega)\,\mathsf{D}_\omega.$$

Moreover given $z \in V$ if J_z denotes the matrix whose entries are null except those corresponding to the row z that are equal to 1, then G^z is the matrix obtained from $\frac{1}{2}\,\mathsf{D}\Big(\mathsf{J}_z\mathsf{R}_{\lambda,\omega}+\mathsf{R}_{\lambda,\omega}\mathsf{J}^*_z-\mathsf{R}_{\lambda,\omega}\Big)\mathsf{D}$ by deleting the row and column corresponding to z.

In particular, when $\lambda = 0$ and $\omega = 1$, then $\mathsf{D}_\omega = \mathsf{I}$ and $\mathsf{S}_\omega = \frac{1}{n}\,\mathsf{J}$ and hence the above corollary becomes the well-known identity

$$\mathsf{L}^\dagger = -\frac{1}{2}\Big[\mathsf{R} - \frac{1}{n}\,[\mathsf{J}\,\mathsf{R} + \mathsf{R}\,\mathsf{J}] + \frac{1}{n^2}\,\mathsf{J}\,\mathsf{R}\,\mathsf{J}\Big],$$

see [11, Theorem 7] and [12, Theorem 3.7]. A similar identity can be found in [1] in the context of spherical Euclidean distance matrices.

The above results allow us to characterize when the Moore-Penrose inverse of $\mathsf{L}_\omega + \lambda(\mathsf{I}-\mathsf{P}_\omega)$ is an M-matrix. Specifically, $\Big(\mathsf{L}_\omega + \lambda(\mathsf{I}-\mathsf{P}_\omega)\Big)^\dagger$ is an M-matrix iff for any $x \neq y$ it is verified that $r_{\lambda,\omega}(x) + r_{\lambda,\omega}(y) \leqslant R_{\lambda,\omega}(x,y)$ or, in an equivalent manner, iff

$$\sum_{z\in V}\Big(R_{\lambda,\omega}(x,z) + R_{\lambda,\omega}(y,z)\Big)\omega^2(z) \leqslant n\,R_{\lambda,\omega}(x,y) + \frac{1}{n}\sum_{t,z\in V} R_{\lambda,\omega}(t,z)\,\omega^2(t)\omega^2(z).$$

The above bound is tight since if we consider the complete graph, $\lambda = 0$ and $\omega = 1$, then $R(x,y) = \frac{2}{n}$ for all $x,y \in V$ and hence the equality happens. In this case, $\mathsf{L} = n\mathsf{I} - \mathsf{J}$ and

$\mathsf{L}^\dagger = \frac{1}{n^2}\,\mathsf{L}$. Moreover, when Γ is a weighted tree, $\lambda = 0$ and $\omega = 1$, it suffices to verify the above condition for adjacent vertices, as was proved in [12]. In fact, in [13] it was proved that for $n \geqslant 5$ this occurs iff Γ is a weighted star.

As a by-product of Proposition 4.4 we can obtain the expression of the effective resistance, $\widehat{R}_{\widehat{\omega}}$, and the total resistance, $\widehat{r}_{\widehat{\omega}}$, of the network $\Gamma_{\lambda,\omega}$, with respect to $\widehat{\omega}$, in terms of $R_{\lambda,\omega}$ and $r_{\lambda,\omega}$ the effective resistance and the total resistance of Γ, with respect to λ and ω. Of course, to do this we must make use of part (iii) of Proposition 3.7.

Corollary 4.6 *We get that* $\widehat{r}_{\widehat{\omega}}(\hat{x}) = \dfrac{n}{\lambda(n+1)^2}$ *and for any* $x, y \in V$ *it is verified that*

$$\widehat{r}_{\widehat{\omega}}(x) = r_{\lambda,\omega}(x) + \frac{1}{\lambda\, n(n+1)^2}, \qquad \widehat{R}_{\widehat{\omega}}(x,\hat{x}) = r_{\lambda,\omega}(x) + \frac{1}{\lambda\, n} \quad \textit{and} \quad \widehat{R}_{\widehat{\omega}}(x,y) = R_{\lambda,\omega}(x,y).$$

Moreover, the Kirchhoff Index of $\Gamma_{\lambda,\omega}$ *with respect to* $\widehat{\omega}$ *is* $\widehat{\mathsf{k}}(\widehat{\omega}) = \dfrac{(n+1)}{n}\,\mathsf{k}(\lambda,\omega) + \dfrac{1}{\lambda}$.

Proof. We know that $\widehat{r}_{\widehat{\omega}}(y) = \dfrac{\widehat{G}(y,y)}{\widehat{\omega}^2(y)}$, for any $y \in V \cup \{\hat{x}\}$. Therefore, by using the expression for $\widehat{G}$ obtained in Proposition 3.7 we get the value for $\widehat{r}_{\widehat{\omega}}(\hat{x})$ and also that for any $x \in V$,

$$\widehat{r}_{\widehat{\omega}}(x) = \frac{G_q(x,x)}{\omega^2(x)} - \frac{n+2}{\lambda(n+1)^2} = r_{\lambda,\omega}(x) + \frac{1}{\lambda\, n} - \frac{n+2}{\lambda(n+1)^2} = r_{\lambda,\omega}(x) + \frac{1}{\lambda\, n(n+1)^2}.$$

On the other hand, Proposition 4.3 establishes that

$$\widehat{R}_{\widehat{\omega}}(x,y) = \frac{\widehat{G}(x,x)}{\widehat{\omega}^2(x)} + \frac{\widehat{G}(y,y)}{\widehat{\omega}^2(y)} - \frac{2\widehat{G}(x,y)}{\widehat{\omega}(x)\widehat{\omega}(y)} \quad \text{for any } x, y \in V \cup \{\hat{x}\},$$

which implies that

$$\widehat{R}_{\widehat{\omega}}(x,\hat{x}) = \frac{G_{\lambda,\omega}(x,x)}{\omega^2(x)} - \frac{(n+2)}{\lambda(n+1)^2} + \frac{n}{\lambda(n+1)^2} + \frac{2}{\lambda(n+1)^2} = r_{\lambda,\omega}(x) + \frac{1}{n\,\lambda} \quad \text{for any } x \in V,$$

and also that

$$\widehat{R}_{\widehat{\omega}}(x,y) = \frac{G_{\lambda,\omega}(x,x)}{\omega^2(x)} + \frac{G_{\lambda,\omega}(y,y)}{\omega^2(y)} - \frac{2G_{\lambda,\omega}(x,y)}{\omega(x)\omega(y)} = R_{\lambda,\omega}(x,y) \quad \text{for any } x, y \in V.$$

Finally, we know that $\widehat{\mathsf{k}}(\widehat{\omega}) = (n+1)\langle \widehat{r}_{\widehat{\omega}}, \widehat{\omega}^2\rangle$ and hence we get that

$$\widehat{\mathsf{k}}(\widehat{\omega}) = (n+1)\widehat{r}_{\widehat{\omega}}(\hat{x}) + (n+1)\langle r_{\lambda,\omega}, \omega^2\rangle + \frac{1}{\lambda(n+1)} = \frac{(n+1)}{n}\,\mathsf{k}(\lambda,\omega) + \frac{1}{\lambda}. \qquad \blacksquare$$

Bearing in mind the above corollary, our main result appears now as a straightforward consequence of Corollary 4.4 and Proposition 3.7.

Theorem 4.7 *Given $\lambda > 0$, $\omega \in \Omega(V)$ and $q = q_\omega + \lambda$, then for any $x, y \in V$ it is verified that*

$$G_q(x,y) = \frac{1}{2}\,\omega(x)\,\omega(y)\,\Big(\widehat{R}_{\widehat{\omega}}(x,\hat{x}) + \widehat{R}_{\widehat{\omega}}(y,\hat{x}) - \widehat{R}_{\widehat{\omega}}(x,y)\Big).$$

We end the paper with the matrix counterpart of the main results in this section. So, we characterize the inverse of any irreducible symmetric M-matrix, singular or not, in terms of the effective resistances of a suitable network, or equivalently, we prove that any irreducible symmetric M-matrix is a resistive inverse.

Theorem 4.8 *Let M be a singular irreducible and symmetric M-matrix of order n and consider $\mathsf{M}^\dagger = (g_{ij})$ its Moore-Penrose inverse. Then there exist a network $\Gamma = (V, c)$ with $|V| = n$ and a weight $\omega \in \Omega(V)$ such that $\mathsf{M} = \mathsf{L}_\omega$. Moreover, if R_{ij}, $i, j = 1, \dots, n$ are the effective resistances of Γ with respect to ω, then*

$$g_{ij} = -\frac{\omega_i \omega_j}{2}\left(R_{ij} - \frac{1}{n}\sum_{k=1}^{n}(R_{ik} + R_{jk})\omega_k^2 + \frac{1}{n^2}\sum_{k,l=1}^{n} R_{kl}\omega_k^2\omega_l^2\right).$$

Theorem 4.9 *Let M be an irreducible Stieltjes matrix of order n and $\mathsf{M}^{-1} = (g_{ij})$ its inverse. Then there exist a network $\Gamma = (V, c)$ with $|V| = n$, a value $\lambda > 0$ and a weight $\omega \in \Omega(V)$ such that $\mathsf{M} = \mathsf{L}_\omega + \lambda\mathsf{I}$. Moreover, if we consider the host network $\Gamma_{\lambda,\omega} = (V \cup \{x_{n+1}\}, \widehat{c}_{\lambda,\omega})$ and $\widehat{R}_{ij}$, $i, j = 1, \dots, n+1$ are the effective resistances of $\Gamma_{\lambda,\omega}$ with respect to $\widehat{\omega}$, then*

$$\mathsf{L}_{\widehat{\omega}} = \begin{bmatrix} \mathsf{M} & -\mathsf{M}\mathsf{w} \\ -\mathsf{w}^*\mathsf{M} & \mathsf{w}^*\mathsf{M}\mathsf{w} \end{bmatrix} = \begin{bmatrix} \mathsf{M} & -\lambda\mathsf{w} \\ -\lambda\mathsf{w}^* & n\lambda \end{bmatrix},$$

where $\mathsf{w} \in \mathbb{R}^n$ is the vector indentifed with ω, and

$$g_{ij} = \frac{\omega_i\omega_j}{2}\left(\widehat{R}_{in+1} + \widehat{R}_{jn+1} - \widehat{R}_{ij}\right), \quad \text{for any } i, j = 1, \dots, n.$$

Since the values $-c_{ij}$, $i \neq j$, are the off-diagonal entries of M, then $\mathsf{M} = \mathsf{L}_\omega + \lambda\mathsf{I}$ is weakly diagonally dominant iff

$$\lambda \geqslant \frac{1}{\omega_i}\sum_{j=1}^{n} c_{ij}(\omega_i - \omega_j), \quad i = 1, \dots, n$$

with strict inequality for at least one index. So, the above theorem generalizes the main result obtained by M. Fiedler in [8] where the inverses of weakly diagonal dominant Stieltjes matrices were characterized.

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